Exercises on moving interfaces in solids

N. Moešs

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Abstract

Six exercises are given in these notes illustrating the extended finite element method (X-FEM), the thick level set model for damage (TLS) and the inequality level set approach (ILS).

1 X-FEM 1:

Goal  Deal with X-FEM stiffness matrices in 1D.

Problem  We consider the following 1D diffusion problem,

\[ u''(x) = 0 \text{ for } x \in \Omega = (0, 1), \; u(0) = 0, \; u(1) = 1, \]

(1)

Defining the space of trial and test functions,

\[ U = \{ u \in V, u(0) = 0, u(1) = 1 \}, \quad U_0 = \{ v \in V, v(0) = v(1) = 0 \}, \]

(2)

where \( V \) is the space with the proper regularity for the solution, we may express the variational form associated to the strong form (9),

\[ u \in U, \quad \int_0^1 u'v' \, dx = 0, \quad \forall v \in U_0. \]

(3)

The domain is discretized using four finite elements of equal size. The finite element approximation, \( u^h \), has three degrees of freedom as well as test functions

\[ \begin{array}{c}
N_1 \\
N_2 \\
N_3 \\
N_4
\end{array} \]

\[ \begin{array}{c}
u = 0 \\
u = 1
\end{array} \]

Figure 1: The 1D model problem with four finite elements.
$u = 0$

$u = 1$

Figure 2: The 1D model problem with a discontinuity located at node 2.

$H(x)$

$H(x)$

Figure 3: Enrichment functions for a discontinuity located at a node (left) and in between two nodes (right).

$v^h$:

\[ u^h(x) = \sum_{i=1}^{3} u_i N_i(x) + N_4(x), \quad v^h(x) = \sum_{i=1}^{3} v_i N_i(x). \]  

(4)

Finite element approximation functions, $N_i$, are depicted in Figure 1.

**Question 1** Assemble the finite element matrices of the problem and check that it returns the exact solution to the problem.

**Question 2** Consider that a crack is placed at a point $x_c$ in the bar. Express the mathematical strong form and weak form of the problem and solve it.

**Question 3** We deal with question 2 but numerically. Consider a crack placed at node 2 (see Figure 2). Build the matrices using a double node approach and a X-FEM approach. Check that the solution obtained in both cases are equal (and exact in this simple 1D case).

**Question 4** Consider now a crack placed on element 2-3 (at 1/3 close to node 2), as depicted in Figure 3 (left). Assemble the X-FEM matrix and check that the exact solution is again recovered.
**Question 5**  For each of the scenarios below, discuss the theoretical model and the X-FEM strategy to deal numerically with the problem (note that the strategy is not unique in each case).

- The crack is now a cohesive zone and there exist a relation between the crack opening and stress on the crack faces.
- Same as above but the material is different on each side of the crack.
- The crack is replaced by a material interfaces perfectly bonded separating different materials.
2 X-FEM 2

Goal

- Understand the relationship between X-FEM and Hansbo type enrichment.
- Understand the impact of cut elements on critical time step in explicit dynamics
- Realize that a naive Lagrange multiplier space to impose Dirichlet boundary conditions with the X-FEM does not work.

Question 1 We consider the Hansbo type enrichment depicted in Figure 4.

- Check that they lead to the same discrete space over the element.
- Rework the question 4 of exercise X-FEM 1 with this other type of enrichment.
- Discuss what needs to be done with the Hansbo functions when the crack crosses a node.

![Figure 4: Comparison between Hansbo and X-FEM enrichment.](image)

Question 2 In explicit dynamics, the time step needs to be small enough to ensure stability of the numerical integration scheme. For linear problem, the maximum allowable time step, $\Delta t_c$, is given by

$$\Delta t_c = \frac{2}{\omega_{\text{max}}}$$  \hspace{1cm} (5)

where $\omega_{\text{max}}$ is the maximum eigenvalue of the system

$$(K - \omega_{\text{max}}^2 M)X = 0$$  \hspace{1cm} (6)

$K$ and $M$ being the mass and stiffness matrices, respectively.
It turns out that the maximum eigenvalue of the global problem is bounded from above by the maximum eigenvalue of all elementary problem taken separately.

\[ \omega^2_{\text{max}} \leq \max_e \omega^2_{\text{max}}, \quad (K^e - \omega^2_{\text{max}} M^e) X^e = 0 \quad (7) \]

Consider the following two questions

- Find the critical time step for a 1D bar element (length \( h \), section: \( S \), density \( \rho \), Young modulus \( E \)), considering a lumped mass matrix.

- The bar element is now cracked in two pieces. Using Hansbo enrichment, show that it is possible to keep the critical time step as before.

\[ \lambda_1 = F, \quad \lambda_2 = F + \frac{[F]}{1-e}, \quad \lambda_3 = F - \frac{[F]}{(1-e)^2} \quad (8) \]

**Figure 5:** Two elements scalar problem.

**Question 3** We consider the grey domain depicted in Figure 5. Set up the equations to find the value of the Lagrange multipliers \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). Check that the result is
where $F = F_1 + F_2$, $[F] = F_1 - F_2$.

Discuss strategies that could be used to avoid these oscillatory Lagrange multipliers.
3 TLS 1:

**Goal:** Obtain analytical solution for the TLS approach on a 1D bar in quasi-static.

**Problem:** We consider a bar depicted in Figure 6. It is clamped on the left side and a displacement $U$ is imposed on the right side. The bar section is $S$. The force needed to impose displacement $U$ is denoted $F$. The evolution is assumed quasi-static.

The local constitutive law is depicted in Figure 7. It corresponds to a free energy

$$\varphi(\epsilon, d) = \frac{1}{2}(1 - d)Ec^2 + h(d), \quad h(d) = Y_c \frac{\alpha d^2}{1 - \alpha d}, \quad 0 \leq \alpha < 1$$  \hspace{1cm} (9)

and the dissipation potential

$$\psi(\dot{d}) = Y_c \dot{d} + I_+ (\dot{d})$$  \hspace{1cm} (10)

where

$$I_+ (d) = \begin{cases} 0 \text{ if } \dot{d} \geq 0 + \infty \text{ if } \dot{d} < 0 \end{cases}$$  \hspace{1cm} (11)

Evolution laws related to the above dissipation potential are

$$\dot{d} \geq 0, \quad Y - Y_c \leq 0, \quad (Y - Y_c) \dot{d} = 0$$  \hspace{1cm} (12)

The objective is to obtain for the TLS regularization of this local model, the characteristic for the force displacement relation $(F, U)$ on the bar.

Regarding the TLS, we assume a linear relation between damage and the level set.

**Question 1** Find the maximum loading that the bar can sustain elastically and discuss why beyond this point the local damage model is physically ill-posed.

![Figure 6: A 1D bar with an evolving damage zone near the origin.](image)

**Question 2** The bar is now undergoing damage. The maximum damage in the bar is denoted $d_0$ and takes place at $x = 0$. Find the relation between the stress and average strain ($\bar{\tau} = U/L$) in the bar with the TLS model. Compare it to the local behavior depicted in Figure 7 for different ration $l_c/L$. 

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First, let us find the analytical solution of the 1D bar problem for an imposed displacement $U$ (or load $F$). The equilibrium $\frac{d\epsilon}{dx} = 0$ yields:

$$\frac{d\epsilon}{dx} = \frac{1}{1 - \frac{\alpha}{\epsilon_c}}$$

and $\epsilon$ is obtained by integrating (12) on the domain:

$$\epsilon(x) = \begin{cases} I & \text{if } 0 \leq x \leq l \\ C_1 & \text{if } l \leq x \leq L \end{cases}$$

Denoting $U$ the end bar displacement and enforcing the displacement continuity over the bar, we obtain the constant:

$$C = \frac{U}{l \log \frac{1}{1 - \frac{\epsilon}{\epsilon_c}}}$$

The load is related to the end displacement through the equivalent stiffness $K_{eq}$:

$$F = \frac{U}{L}$$

Using the linear relation $\frac{d\epsilon}{d\sigma}$, $K_{eq}$ reads:

$$K_{eq} = \frac{E}{1 - \frac{\epsilon}{\epsilon_c} \log \frac{1}{1 - \frac{\epsilon}{\epsilon_c}}}$$

Therefore, the constant $C$ may be expressed as a simpler function of the load:

$$C = \frac{F}{E}$$

Question 3 Instead of a linear damage profile, consider a parabolic profile

$$d = 1 - (1 - \frac{\epsilon}{\epsilon_c})^2$$

or cubic profile

$$d = 1 - (1 - \frac{\epsilon}{\epsilon_c})^3$$

Analyze how the above curves are modified using an asymptotic expansion for $d_0 \to 0$. 

Figure 7: Local stress strain relationship for a growing strain.
4 TLS 2:

**Goal** Understand the relationship between cohesive zone model (CZM) and TLS model in a 1D case.

**Problem** A cohesive zone model and a TLS model are considered and we analyze conditions under which both models yield the same force displacement curve in a bar.

- For the local damage model, we take the one given in section TLS1 with \( \alpha = 0 \) and a given \( d(\phi) \) TLS relation.
- The cohesive zone model is given by the stress \( \sigma \) to the opening \( w \) relation.

**Question 1** Expressing the fact that for a given stress in the bar, both model must yield the same displacement, find a first relation tying both models.

**Question 2** Find the relation between the critical stress \( \sigma_c \) in the CZM and the quantity \( Y_c \). Using the non-dimensional following quantities

\[
\tilde{\sigma} = \frac{\sigma}{\sqrt{2EY_c}}, \quad \tilde{w} = \frac{w}{l_c \sqrt{2Y_c/E}}, \quad \tilde{l} = \frac{l}{l_c}, \quad \tilde{\phi} = \frac{\phi}{l_c}
\]

rewrite the result of question 1.

**Question 3** Starting from a TLS model, find the corresponding CZM model. Imagine how a simple code could be built to make this transfer.

**Question 4** Discuss graphically the equivalence of dissipation in both models. The, using the following non-dimensional quantities

\[
\hat{\sigma} = \frac{\sigma}{\sigma_c}, \quad \hat{w} = \frac{w}{w_c}, \quad \hat{l} = \frac{\sigma_c w_c E}{l}
\]

show that the dissipation equivalence reduces to an area equivalence.

**Question 5** Using results of question 4, starting from a CZM model, find the corresponding TLS model. As an application, consider \( \hat{\sigma} = 1 - \sqrt{\hat{w}} \). Imagine how a simple numerical code could be built to make this transfer for more complex relations.
5 ILS 1: Limit of extensibility

Goal: Learn

- the mathematical formulation of a constitutive law involving inequality.
- the concept of configurational force applied to this inequality context.
- how to take derivative of a variational formulation and make directional derivative appear.

Problem A bar of length $L$, section $S$ and Young modulus $E$ is subjected to a uniform volumic force $f$ acting to the right. The left of the bar is clamped and no forces are applied on its right side. The bar is made out of a material whose strain $\epsilon$ may not go beyond $\alpha$. We deal with small deformation assumption.

Question 1 Give the equation that must fulfill the displacement field $u$ and find the analytical solution to the problem. Denote $l$ as the extent of the zone where the constraint is active. Observe the regularity of the solution across $l$. Observe also the relation between the stress and strain in the active zone.

Question 2 Consider now that $l$ are $f$ are given but that they do not correspond to each other according to the exact solution. Define an approximate solution in this case and explain which equation you have relaxed from the initial problem.

Question 3 Express the potential energy of the approximate solution and take its derivative with respect to $l$. What is the property of the potential energy when the approximate solution is the exact one?

Question 4

- Formulate the approximate problem as a variational problem including a Lagrange multiplier (denoted $p$ as pressure) to enforce $\epsilon = \alpha$ on $[0, l]$.
- Take the derivative of this problem to make the directional derivative of the unknowns appear. The directional derivative will use a piecewise linear configurational velocity $w(x)$ on the bar.
- Solve the above problem and check that it gives the correct directional derivative of $p$ at $l$.

Question 5 How would you implement the strategy of following the evolution of the active zone with the X-FEM and level sets?
6 ILS 2: contact of a string on a flat obstacle

Goal
Learn
• proper boundary conditions on the boundary of a contact zone
• how to compute directional derivative of a potential energy
• relation between adhesion and surface energy (and its similarity with fracture energy).

Definition of the problem
In this exercise, we will focus on an elastic string fixed at both extremities. For the sake of simplicity, we will only consider the case of small displacements. This string is loaded with an homogeneous pressure, and a rigid plane is positioned underneath, parallel to the string. As the problem is symmetric, we will examine half of the string (cf. Figure 8). Throughout the exercise, the following notation will be used:

• $x$ the coordinate of a point on the string,
• $u(x)$ the vertical displacement of this point,
• $T$ the tension in the string,
• $p > 0$ the homogeneous pressure applied to the string,
• $d > 0$ the signed distance from the string to the rigid plane.
• $L$ the half-length of the string

In this problem $T$ will be set constant throughout the string.

Figure 8: Elastic string bounded by a plane surface

To find the equilibrium equation of the problem without contact we:
• isolate a piece of string of length $dx$,

• apply the second Newton’s law to the string projected onto the $y$-axis (we will name $\alpha$ the angle of the string with the horizontal),

• approximate $\cos(\alpha)$ by 1 and $\sin(\alpha)$ by $\frac{du}{dx}$,

• have $dx$ to tend to 0.

The equation is then:

$$T \frac{d^2 u}{dx^2} = p$$  \hspace{1cm} (17)

If a pointwise force $\lambda > 0$ acting downward is applied at $x_f$ the same method give us the following relation:

$$T \left[ \frac{du}{dx} \right]_{x_f} = \lambda$$  \hspace{1cm} (18)

**Question 1** Find the exact solution to the problem. (hint: The contact zone is $[l, L]$, find $l$). Discuss its regularity.

**Question 2** Consider now that the contact zone (i.e. $l$) is imposed independently of the loading $p$. This problem is the so-called approximate problem. Discuss the meaning of this problem and see how it can be solved. Discuss the type of reaction forces obtained.

**Question 3** Write the approximate problem in variational format and express the change in solution as $l$ is changing. Imagine a numerical scheme to iteratively find the exact solution.

**Question 4** Find a way to include adhesion in the problem. (Hint: the adhesive force is given by $\sqrt{2T\gamma}$ where $\gamma$ is the surface tension.

**Question 5** Show that the potential energy is stationary for the exact $l$ value and discuss it’s property with and without adhesion.